TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 348, Number 1, January 1996

ON THE VARIANCES OF OCCUPATION TIMES OF CONDITIONED BROWNIAN MOTION

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ABSTRACT. We extend some bounds on the variance of the lifetime of two–dimensional Brownian motion, conditioned to exit a planar domain at a given point, to certain domains in higher dimensions. We also give a short "analytic" proof of some existing results.

1. INTRODUCTION

This paper studies questions related to the variance of the lifetime of certain h-processes. Our estimates are related to a result of B. Davis, stated below (Theorem 1.1), and in fact we give a short, "analytic" proof of this result. h-processes are intimately connected with many aspects of partial differential equations and harmonic analysis, and in particular variance estimates have recently been used to study intrinsic ultracontractivity, in [1] and [4].

If A is a Borel subset of $\mathbb{R}^d, d \geq 2$, the Lebesgue measure, closure, complement, and Euclidean boundary of A are respectively denoted by $|A|, \overline{A}, A^c$, and ∂A. Let D be a domain of $\mathbb{R}^d$, which has a Green function, let P_x and E_x be probability and expectation of standard d-dimensional Brownian motion started at x, and let P_x^y and E_x^y denote the probability and expectation of this motion either conditioned to exit $D \setminus \{y\}$ at y, if $y \in D$, or conditioned to exit D at the point y in its minimal Martin boundary Δ, if $y \in \Delta$. Formally, these are the h-processes with h respectively the Green function of D, denoted by $G(\cdot, y)$, or the Martin kernel function of D, denoted by $K(\cdot, y)$. These are the basic h-processes in the sense that all the other h-processes are mixtures of them, see [3] as a reference. We will discuss h-processes in more detail later. We use τ_D to designate the first exit time of a process from D, and often shorten τ_D to τ. Positive constants c, C, c_M, C_M may depend on the dimension and are not necessarily the same at each occurrence. The letters x and y are used to designate respectively the starting and exit points of motions.

By a cube Q of $\mathbb{R}^d, d \geq 2$, we always mean a closed cube. A Whitney decomposition of D, denoted by $W(D) = \{Q_i\}_{i \geq 0}$, is a collection of closed cubes in D with disjoint interiors, with union D, satisfying, for all i,

$$1 \leq \frac{d(Q_i, \partial D)}{\ell(Q_i)} \leq 4\sqrt{2}.$$

Received by the editors October 24, 1994.
1991 *Mathematics Subject Classification.* Primary 60J65, 60J05.
Key words and phrases. Conditioned Brownian motion, h-processes.

See [10] for a proof that all domains have a Whitney decomposition. Here $\ell(Q_i)$ is the side length of Q_i and $d(\cdot,\cdot)$ is the Euclidean distance between points, sets, or points and sets. The Whitney decomposition gives rise to a quasi–hyperbolic distance in D in the following way. Fix $Q, Q' \in W(D)$. We say that $Q = Q_0 \to Q_1 \to \dots \to Q_m = Q'$ is a Whitney chain connecting Q and Q' of length m if $Q_i \in W(D)$ for all i and if $Q_{i-1} \cap Q_i \neq \varnothing$, $1 \leq i \leq m$. Define the Whitney distance $\rho_D(Q, Q')$, or briefly $\rho(Q, Q')$, to be the length of the shortest Whitney chain connecting Q and Q'. See [8] for a reference. If $Q \in W(D)$, we denote $\int_0^\tau I(Z_t \in Q)dt$, the total time Z_t spends in Q, by T_Q, and let $P_Q = P_x^y(\tau_{Q^c} < \tau)$ stand for the probability Z_t ever hits Q, where τ_{Q^c} is the first time Z_t hits Q. By $\operatorname{cov}(T_{Q_i}, T_{Q_j}) = \operatorname{cov}_x^y(T_{Q_i}, T_{Q_j})$ we mean the covariance of T_{Q_i} and T_{Q_j} with respect to P_x^y. The following result is due to Burgess Davis [3].

Theorem 1.1. *If Q and R are Whitney cubes of a Whitney decomposition of a simply connected planar domain D, then if $x \in D$ and $y \in \Delta$,*

$$|cov(T_Q, T_R)| \leq Ce^{-c\rho(Q,R)}|Q||R|(P_Q + P_R).$$

Davis' proof of this theorem has a substantial probabilistic component and is somewhat involved. Here we first prove the following formula for $\operatorname{cov}(T_Q, T_R)$ in terms of K and G, for $y \in \Delta$. Let Q and R be subdomains of D which have disjoint interiors. Then

(1.1)

$$\begin{aligned}\operatorname{cov}(T_Q, T_R) = \int_Q \int_R \bigg\{ G(q,r) \left[G(x,r) \frac{K(q,y)}{K(x,y)} + G(x,q) \frac{K(r,y)}{K(x,y)} \right] \\ - G(x,r)G(x,q) \frac{K(q,y)K(r,y)}{K(x,y)^2} \bigg\} dqdr, \\ x \in D,\ y \in \Delta.\end{aligned}$$

We use this formula to prove Theorem 1.1 as well as the following theorem, for domains in $\mathbb{R}^d, d \geq 2$, of the form

$$D = D_f = \{x = (x_1, \dots, x_d) \colon x_d > f(x_1, \dots, x_{d-1})\},$$

where f is a Lipschitz function on $\mathbb{R}^{d-1}$. We let $M(f) = M$ stand for the Lipschitz constant of f.

Theorem 1.2. *If Q and R are Whitney cubes of $D = D_f$ with disjoint interiors, then*

$$|cov(T_Q, T_R)| \leq C_M e^{-c_M \rho(Q,R)} |Q|^{2/d} |R|^{2/d} (P_Q + P_R).$$

Clearly, Theorem 1.1 can be used in some cases to bound the variance of $\sum T_{\Gamma_i} = \int_0^\tau I(Z_s \in \bigcup \Gamma_i)ds$, using the formula,

$$\operatorname{var}(\sum T_{\Gamma_i}) = \sum \operatorname{var} T_{\Gamma_i} + \sum_{i \neq j} \operatorname{cov} (T_{\Gamma_i}, T_{\Gamma_j}),$$

where Γ_i are Whitney cubes with disjoint interiors. See [1], [3], and [4] for examples. Theorem 1.2 can be similarly employed. Our proofs, unchanged, prove the analogous theorems about Brownian motion conditioned to exit a domain minus a point at that point. See [3] for a description of these processes. We then indicate in a brief paragraph how this proof can be modified to prove Theorem 1.1. See [7] for a proof that the Martin boundary of D_f is the Euclidean boundary of D_f.

2. Covariance formula

In this section, (1.1) and some lemmas will be proved. The σ–fields of a process $Z_t,\ t \geq 0$, are denoted by $\mathcal{F}(u) = \sigma(Z_s, s \leq u)$. If g is a positive harmonic function on Γ, a domain of $\mathbb{R}^d$, $d \geq 2$, then the h-process in Γ associated with g is determined by the following transition density function:

$$p_t^g(x,y) = \frac{g(y)}{g(x)}\ p_t(x,y). \tag{2.1}$$

The corresponding probability and expectation are denoted by P_x^g and E_x^g respectively. See [6] for more information on h-processes. Here we recall that an h-process is a strong Markov process with continuous paths up to its lifetime τ_Γ, and if η is a stopping time of this process and $A \in \mathcal{F}(\eta)$, then

$$P_x^g(A \cap \{\eta < \tau_\Gamma\}) = \int_{A\cap\{\eta<\tau_\Gamma\}} \frac{g(Z_\eta)}{g(x)}\ dP_x. \tag{2.2}$$

We define, for any real number a,

$$D_a = \{|x_i| < 2^a,\ 1 \leq i \leq d\} \cap D \text{ and}$$
$$L_a = \{|x_i| = 2^a,\ 1 \leq i \leq d\} \cap D.$$

And we always assume that $0 \in \partial D$ which is the center of our "rectangles" D_a. It is easy to see that all D_a are simply connected Lipschitz domains with Lipschitz constants that can be bounded by a number that depends only on $M > 0$, the Lipschitz constant of f, that will be denoted by M again. Let $\tau_a = \inf\{t\colon Z_t \in L_a\}$ be the first time Z_t hits L_a. By $\mathrm{cov}_x^g(T_Q, T_R)$ we mean the covariance of T_Q and T_R with respect to P_x^g.

Theorem 2.1. *Let g be a positive harmonic function on a Greenian domain $\Gamma \subseteq \mathbb{R}^d$, $d \geq 2$, and Q and R be subdomains of Γ having disjoint interiors, then*

$$cov_x^g(T_Q,T_R) = \int_Q\int_R \Big\{ G_\Gamma(q,r)\Big[G_\Gamma(x,r)\frac{g(q)}{g(x)} + G_\Gamma(x,q)\frac{g(r)}{g(x)}\Big] - G_\Gamma(x,r)G_\Gamma(x,q)\frac{g(q)g(r)}{g(x)^2}\Big\} dqdr.$$

Proof. By (2.1),

$$\begin{aligned} E_x^g T_Q &= E_x^g \int_0^{\tau_\Gamma} I(Z_t \in Q)dt \\ &= \int_0^\infty \int_Q p_t^g(x,q)dqdt \\ &= \int_Q G_\Gamma(x,q)\frac{g(q)}{g(x)}\ dq. \end{aligned}$$

Similarly,

$$E_x^g T_R = \int_R G_\Gamma(x,r)\ \frac{g(r)}{g(x)}\ dr.$$

Since

$$
\begin{aligned}
E_x^g T_Q T_R &= E_x^g \int_0^{\tau_\Gamma} \int_0^{\tau_\Gamma} I(Z_t \in Q) I(Z_s \in R)\ dtds \\
&= \int_0^\infty \int_0^\infty P_x^g(Z_t \in Q, Z_s \in R, t < \tau_\Gamma, s < \tau_\Gamma)\ dsdt \\
&= \iint_{s \le t} \left[\int_Q \int_R p_s^g(x,r) p_{t-s}^g(r,q) dq dr \right] dsdt \\
&\quad + \iint_{s>t} \left[\int_Q \int_R p_t^g(x,q) p_{s-t}^g(q,r) dq dr \right] dsdt \\
&= \int_Q \int_R \left\{ \int_0^\infty \left[\int_s^\infty p_{t-s}^g(r,q) dt \right] p_s^g(x,r) ds \right\} dqdr \\
&\quad + \int_Q \int_R \left\{ \int_0^\infty \left[\int_t^\infty p_{s-t}^g(q,r) ds \right] p_t^g(x,q) dt \right\} dqdr \\
&= \int_Q \int_R G_\Gamma(r,q) \frac{g(q)}{g(r)} G_\Gamma(x,r) \frac{g(r)}{g(x)} dqdr \\
&\quad + \int_Q \int_R G_\Gamma(q,r) \frac{g(r)}{g(q)} G_\Gamma(x,q) \frac{g(q)}{g(x)} dqdr \\
&= \int_Q \int_R G_\Gamma(q,r) \left[G_\Gamma(x,r) \frac{g(q)}{g(x)} + G_\Gamma(x,q) \frac{g(r)}{g(x)} \right]\ dqdr,
\end{aligned}
$$

the theorem follows easily from

$$
\operatorname{cov}_x^g(T_Q, T_R) = E_x^g T_Q T_R - E_x^g T_Q E_x^g T_R.
$$

□

Taking $g(x) = K(x,y)$ in Theorem 2.1, for $y \in \Delta$, gives (1.1).

We employ the boundary Harnack principle for Lipschitz domains in the proof of the next lemma. See Jerison and Kenig [9] for a statement of this principle. We will use not only this principle but also the following consequence. We let x_0 be the reference point of our kernel functions such that $d(x_0, \partial D_f) \ge \frac{1}{M}$.

Lemma 2.0. *Let u and v be positive and harmonic functions in D_2. Suppose there are positive constants c and C such that*

$$
c < \frac{u(r)}{v(r)} < C,
$$

where r is the point of ∂D_0 directly above y. Suppose that u and v have boundary limits 0 at each point of $\partial D_f \cap \partial D_2$ except y. Then there are constants c_M and C_M which depend only on c, C, and M, such that

$$
c_M < \frac{u(z)}{v(z)} < C_M, \qquad z \in \partial D_1 \setminus \partial D_f. \tag{2.3}
$$

Proof. For points z not too close to ∂D_f, we use the Harnack inequality. The truth of (2.3) for these points, together with the boundary Harnack principle, gives its truth for all $z \in \partial D_1 \setminus \partial D_f$. □

Lemma 2.1. *There exists an integer $n_0 = n_0(M)$, such that*

(i) *for any x and y inside L_0 and for any positive integer m, we have*

$$P_x^y(\tau_{mn_0} < \tau) \leq \frac{1}{2^m};$$

(ii) *for x and y outside L_{mn_0}, we have*

$$P_x^y(\tau_0 < \tau) \leq \frac{1}{2^m}.$$

Proof. Proof of (i):

It is enough to show that there exists an integer $n_0 = n_0(M)$ such that

$$P_x^y(\tau_{n_0} < \tau) \leq \frac{1}{2}, \text{ for any } x \text{ and } y \text{ inside } L_0, \tag{2.4}$$

since the general case follows from this by iteration and scaling.

Let K_2 be the kernel for D_2. We have by Lemma 2.0, with

$$u(x) = \frac{K_2(x,z)}{K_2(r,z)} \text{ and } v(x) = \frac{K(x,y)}{K(r,y)},$$

where z is a fixed point of L_2, that

$$\frac{1}{M}\frac{K_2(x,z)}{K(x,y)} \leq \frac{K_2(r,z)}{K(r,y)} \leq M\frac{K_2(x,z)}{K(x,y)}, x \in \partial D_1 \setminus \partial D_f. \tag{2.5}$$

Furthermore, if ω_z denotes harmonic measure on ∂D_2 with respect to D_2 and if $x \in \partial D_1 \setminus \partial D_f$ and $j \geq 2$, Lemma 2.0 gives

$$\begin{aligned}
P_x^y(\text{ ever hit } L_j) &= \int_{L_2} P_z^y(\text{ ever hit } L_j)\frac{K(z,y)}{K(x,y)}d\omega_x(z) \\
&= \int_{L_2} P_z^y(\text{ ever hit } L_j)K(z,y)\frac{K_2(x,z)}{K(x,y)}d\omega_{x_0}(z) \\
&\leq M\int_{L_2} P_z^y(\text{ ever hit } L_j)K(z,y)\frac{K_2(r,z)}{K(r,y)}d\omega_{x_0}(z) \\
&\leq M\int_{L_2} P_z^y(\text{ ever hit } L_j)\frac{K(z,y)}{K(r,y)}d\omega_r(z) \\
&= MP_r^y(\text{ ever hit } L_j). \qquad (2.6)
\end{aligned}$$

Furthermore, we note that (2.6) for $x \in \partial D_1 \setminus \partial D_f$ implies (2.6) for $x \in D_1$, by a simple conditioning argument. Thus, to prove (i), it suffices to show that (i) holds for $x = r$. We claim

$$\frac{K(z,y)}{K(r,y)} < M, \qquad z \in \partial D_1 \setminus \partial D_f. \tag{2.7}$$

This follows from Lemma 2.0 applied to $u(z) = \frac{K(z,y)}{K(r,y)}$ (note $K(r,y) \le MK(x_0,y)$ by the Harnack inequality), and the function $v(z) = P_z(B_{\tau_{D_f}} \in \partial D_f \setminus \partial D_2)$. It is easy to show $v(r) > \frac{1}{M} > 0$. Now (2.7) implies

$$\frac{K(z,y)}{K(r,y)} < M, \text{ if } z \in D_f \setminus D_1, \tag{2.8}$$

and it is easily shown that $\omega_r^j(L_j) \le \theta_M^{j-1}$, where $\theta_M < 1$ and ω^j is the harmonic measure on ∂D_j with respect to D_j, (see Davis and Zhang [5] e.g.) since this is easily shown if D_f is a cone with vertex y. Thus

$$\begin{aligned} P_r^y(\text{ ever hit } L_j) &\le \int_{L_j} \frac{K(z,y)}{K(r,y)} d\omega_r^j(z) \\ &\le M\theta_M^{j-1}, \end{aligned}$$

which implies (i) for $x = r$.

Proof of (ii):

By scaling, it is equivalent to show that, for x and y outside L_0,

$$P_x^y(\tau_{-mn_0} < \tau) \le \frac{1}{2^m}.$$

We can use the same argument as that of the proof of (i) to show this. □

The two–dimensional case of the following lemma is due to Burgess Davis [3], see also M. Cranston [2]. Here we only sketch an "analytic" proof for dimensions three and higher.

Lemma 2.2. *Let Γ be a domain of $\mathbb{R}^d$, $d \ge 2$, and Q be a Whitney cube of Γ, then*

$$c|Q|^{2/d} P_Q \le E_x^y T_Q \le C|Q|^{2/d} P_Q.$$

Proof. For $z \in Q$, Harnack's inequality applied to $K(\cdot, y)$ implies that

$$\begin{aligned} E_z^y T_Q &= \int_Q G_\Gamma(z,q) \frac{K_\Gamma(q,y)}{K_\Gamma(z,y)} \, dq \\ &\le C \int_Q G_\Gamma(z,q) dq \\ &\le C \int_{B(z,\sqrt{d}\ell(Q))} \frac{1}{|z-q|^{d-2}} \, dq \\ &\le C|Q|^{2/d}, \end{aligned}$$

where $B(z,r)$ is the ball of radius r centered at z. Thus, using the strong Markov property at the time Q is hit, we get

$$E_x^y T_Q \le C|Q|^{2/d} P_Q.$$

If we let λQ be the scaling of Q by λ with respect to the center of Q, and let P^{2Q} and E^{2Q} stand for probability and expectation of Brownian motion killed at $\partial(2Q)$, since $G_{2Q} \le G_\Gamma$, we obtain, for $z \in \partial Q$,

$$\begin{aligned} E_z^y T_Q &\ge c \int_Q G_{2Q}(z,q)dq \\ &= cE_z^{2Q} T_Q. \end{aligned}$$

Since $P_z^{2Q}(\tau_{(\frac{1}{2}Q)^c} < \tau_{(2Q)^c}) > c > 0$, if $z \in \partial Q$,

$$\begin{aligned} E_z^{2Q} T_Q &\ge c \inf_{w \in \partial(\frac{1}{2}Q)} \{E_w^{2Q} T_Q\} \\ &\ge c \inf_{w \in \partial(\frac{1}{2}Q)} \{E_w^{2Q} \tau_{B(w, \frac{1}{4}\ell(Q))}\} \\ &\ge c|Q|^{2/d}. \end{aligned}$$

Again, the strong Markov property at the time Q is hit implies that

$$E_x^y T_Q \ge c|Q|^{2/d} P_Q.$$

□

Let Q and R be two Whitney cubes of D and let $\lambda > 0$. By scaling, if we use $\lambda Q_i,\ i \ge 0$, to partition λD, where $Q_i,\ i \ge 0$, partition D and λD is the usual scaling D by λ, we have $\rho_{\lambda D}(\lambda Q, \lambda R) = \rho_D(Q,R)$ and $\mathrm{cov}_{\lambda x}^{\lambda y}(T_{\lambda Q}, T_{\lambda R}) = \lambda^4 \mathrm{cov}_x^y(T_Q, T_R)$. Thus we may and do assume from now on by scaling again, without loss of generality, that $\ell(R) \ge \ell(Q) = \frac{1}{M}$, where $\frac{1}{M}$ is a small positive constant depending only on M, and that $d(Q, \partial D) = c_M d(Q, q_0)$, where $c_M > 0$ and depends only on M and $q_0 \in \partial D$, just below the center of Q. Again, we may assume that $q_0 = 0$. Otherwise, we will let our "rectangles" L_a be centered at q_0.

Lemma 2.3. *There exists a positive constant C_M such that if $\rho(Q,R) \ge C_M k$, for a positive integer k, then*

$$d(Q,R) \ge 2^{10k}.$$

Proof. If $d(Q,R) < 2^{10k}$, then it is not hard to show that there exists a Whitney chain connecting Q and R of length less than $C_M k$, for some $C_M > 0$, all cubes of which lie in two cones of D containing Q and R respectively, with vertices at the boundary of D, and with apertures $\theta_M > 0$. It is not hard to argue that the natural chain thus constructed has at most $C_M k$ cubes, so that $\rho(Q,R) \le C_M k$.□

We set $L'_a = L_{n_0 a}, D'_a = D_{n_0 a}$. Now by the assumptions on Q that $\ell(Q) = \frac{1}{M}$ and $d(Q, \partial D) = c_M d(Q, 0)$, we know that Q is inside L'_3, and it follows immediately from this lemma that if $\rho(Q,R) \ge C_M k$, for a large positive integer k and some constant C_M being the product of the constant of Lemma 2.3 and n_0, then R is outside L'_{9k}. For any $x \in D$ and $y \in \partial D$ fixed, it is easy to see that there is a number $s_0 > 0$ such that Q is inside L'_{s_0} and R is outside L'_{s_0+2k}, and furthermore neither x nor y belongs to that part of D lying between L'_{s_0} and L'_{s_0+2k}. By scaling, we may assume that $s_0 = 0$. The following lemma comes from [9].

Lemma 2.4. *Let $\delta > 2$, μ be a positive finite measure on a set S, and θ be a measurable function on S such that $0 < a \le \theta \le A$. If we denote*

$$B(\theta) = \sup\left\{\int_S \theta(s)W(s)d\mu(s)/\int_S W(s)d\mu(s) \colon \delta^{-1} < W < \delta\right\},$$
$$b(\theta) = \inf\left\{\int_S \theta(s)W(s)d\mu(s)/\int_S W(s)d\mu(s) \colon \delta^{-1} < W < \delta\right\},$$

then $\dfrac{B(\theta)}{b(\theta)} - 1 \le (1 - \frac{1}{4}\delta^{-2})(\frac{A}{a} - 1)$.

Let $\Omega_j = D'_{2k-j}, S_j = L'_{2k-j}$, for $j = 1, 2, \dots, k$. Let K^j and ω_j be respectively the Martin kernel function and harmonic measure for Ω_j at the fixed reference point x_0 in D_0. Then we have the following lemma. As earlier, we sometimes use M to stand for any constant depending only on the Lipschitz constant M of f.

Lemma 2.5. *For any $j \in \{1, 2, \dots, k\}$, any $x \in \Omega_{j+1}$, and any $z, \tilde{z} \in S_j$, we have*

$$M^{-1} \le \frac{K^j(x,z)}{K^j(x,\tilde{z})} \le M.$$

Proof. First for $z, \tilde{z} \in S_j$ such that $d(z, \tilde{z}) \le d(x, z)$, by Theorem 5.20 of [9], we have

$$M^{-1} \le \frac{K^j(x,z)}{K^j(x,\tilde{z})} \le M.$$

Now there is an absolute constant N such that there are N points $z_1 = z,\ z_2, \dots,$ $z_N = \tilde{z}$ of S_j and for each pair z_i, z_{i+1} we have $d(z_i, z_{i+1}) \le d(z_i, x),\ i = 1, 2, \dots,$ $N - 1$, therefore

$$M^{-1} \le \frac{K^j(x,z_i)}{K^j(x,z_{i+1})} \le M, \qquad i = 1, 2, \dots, N-1.$$

Thus

$$M^{-N} \le \frac{K^j(x,z)}{K^j(x,\tilde{z})} = \prod_{i=1}^{N-1} \frac{K^j(x,z_i)}{K^j(x,z_{i+1})} \le M^N.$$

Denote M^N by M again, since it only depends on M. We are done. □

Lemma 2.6. *There exists a positive constant $\eta = \eta_M < 1$ such that for any x, q inside L_0 and $y \in \partial D$, y, r outside L'_{2k} if we let $H(x,z) = H_y(x,z) = G(x,z)\frac{K(z,y)}{K(x,y)}$, $z \in D$, then we have*

$$\left|\frac{H(x,r)}{H(q,r)} - 1\right| \le C \cdot \eta^k.$$

Proof. For any $j \in \{1, 2, \dots, k\}$, let $\tilde{z} \in S_j$ be fixed and for any $z \in S_j$, we define

$$W(z) = W^j(z) = \frac{K^j(x,z)}{K^j(x,\tilde{z})},$$
$$\theta(z) = \frac{H(z,r)}{H(q,r)}, \text{ and}$$
$$d\mu(z) = K(z,y)d\omega_j(z).$$

Now the boundary Harnack principle [9] implies that

$$M^{-1} < \theta(z) < M, \quad \text{for } z \in S_j.$$

Lemma 2.5 tells us that $M^{-1} < W(z) < M$, for $z \in S_j$. Since G, K are harmonic, we have

$$\begin{aligned} G(x,r) &= \int_{S_j} K^j(x,z) G(z,r) d\omega_j(z) \text{ and} \\ K(x,y) &= \int_{S_j} K^j(x,z) K(z,y) d\omega_j(z). \end{aligned}$$

Thus,

$$\frac{H(x,r)}{H(q,r)} = \frac{\int_{S_j} W(z)\theta(z) d\mu(z)}{\int_{S_j} W(z) d\mu(z)}.$$

If we let

$$\begin{aligned} b_j &= \inf \{\theta(z) \colon z \in S_j\} \text{ and} \\ B_j &= \sup \{\theta(z) \colon z \in S_j\}, \end{aligned}$$

then

$$b_{j+1} \geq \inf \left\{ \frac{\int_{S_j} \theta(z) F(z) d\mu(z)}{\int_{S_j} F(z) d\mu(z)} \colon M^{-1} < F < M \right\}$$

and

$$B_{j+1} \leq \sup \left\{ \frac{\int_{S_j} \theta(z) F(z) d\mu(z)}{\int_{S_j} F(z) d\mu(z)} \colon M^{-1} < F < M \right\}.$$

Lemma 2.4 implies that

$$\frac{B_{j+1}}{b_{j+1}} - 1 \leq (1 - \frac{1}{4} M^{-2})(\frac{B_j}{b_j} - 1).$$

Thus,

$$\left| \frac{B_j}{b_j} - 1 \right| \leq C^2 (1 - \frac{1}{4} M^{-2})^j.$$

Let $\eta_M = 1 - \frac{1}{4} M^{-2}$. We have, by the maximal principle,

$$\left| \frac{H(z,r)}{H(q,r)} - 1 \right| \leq C \cdot \eta^j, \text{ for } z \in \Omega_j, \ j = 1, \dots, k.$$

Therefore

$$\left| \frac{H(x,r)}{H(q,r)} - 1 \right| \leq C \cdot \eta^k.$$

□

3. Covariance estimates

The following proposition essentially handles the case where $\rho(Q,R)$ is small.

Proposition 3.1. *If Q and R are Whitney cubes of D with disjoint interiors, then*

$$|\operatorname{cov}(T_Q,T_R)| \le C|Q|^{2/d}|R|^{2/d}(P_Q+P_R). \tag{3.1}$$

Proof. If we let

$$I = \int_Q\int_R G(q,r)G(x,r)\frac{K(q,y)}{K(x,y)}\ dqdr,$$

$$II = \int_Q\int_R G(q,r)G(x,q)\frac{K(r,y)}{K(x,y)}\ dqdr,$$

and

$$III = \int_Q\int_R G(x,q)G(x,r)\frac{K(q,y)K(r,y)}{K(x,y)^2}\ dqdr,$$

then (1.1) implies that

$$\operatorname{cov}(T_Q,T_R) = I + II - III.$$

Together with the fact that $P_R\cdot\max_{r\in R}P_r^y(Q)\le CP_Q$, Lemma 2.2 implies that

$$\begin{aligned} I &= \int_R G(x,r)\frac{K(r,y)}{K(x,y)}\left(\int_Q G(q,r)\frac{K(q,y)}{K(r,y)}\ dq\right)dr \\ &\le C|R|^{2/d}P_R(\max_{r\in R}P_r^y(Q)|Q|^{2/d}) \\ &\le C|Q|^{2/d}|R|^{2/d}(P_Q+P_R), \\ II &= \int_Q G(x,q)\frac{K(q,y)}{K(x,y)}\left(\int_R G(q,r)\frac{K(r,y)}{K(q,y)}\ dr\right)dq \\ &\le C|Q|^{2/d}P_Q(\max_{q\in Q}P_q^y(R)|R|^{2/d}) \\ &\le C|Q|^{2/d}|R|^{2/d}(P_Q+P_R), \end{aligned}$$

and

$$\begin{aligned} III &= \int_Q G(x,q)\frac{K(q,y)}{K(x,y)}\ dq\int_R G(x,r)\frac{K(r,y)}{K(x,y)}\ dr \\ &\le C|Q|^{2/d}|R|^{2/d}P_Q\cdot P_R \\ &\le C|Q|^{2/d}|R|^{2/d}(P_Q+P_R). \end{aligned}$$

Therefore, $|\operatorname{cov}(T_Q,T_R)|\le C|Q|^{2/d}|R|^{2/d}(P_Q+P_R)$. □

If Q and R have common interiors, then $Q=R$ for $Q,\ R\in W(D)$. The proof of Proposition 3.1 implies that $E_x^yT_Q^p\le C_p(|Q|^{2/d})^p\cdot P_Q$, for $p>0$.

For $p=2$, we have $E_x^yT_Q^2\le C(|Q|^{2/d})^2\cdot P_Q$, so

$$\begin{aligned} |\operatorname{cov}(T_Q,T_Q)| &\le (E_x^yT_Q^2)+(E_x^yT_Q)^2 \\ &\le C|Q|^{2/d}\cdot|Q|^{2/d}(P_Q+P_Q) \\ &= C(|Q|^{2/d})^2\cdot P_Q. \end{aligned}$$

With the following proposition that essentially handles the case where $\rho(Q,R)$ is large, we can complete the proof of Theorem 1.2.

Proposition 3.2. *If Q and R are Whitney cubes of D with disjoint interiors, satisfying $\rho(Q,R) \ge C_M \cdot m$, where C_M is the constant in Lemma 2.3 times n_0 and m is a large integer, then*

$$|cov(T_Q, T_R)| \le C_M e^{-C_M \rho(Q,R)} |Q|^{2/d} |R|^{2/d} (P_Q + P_R).$$

Proof. Using the same ideas that inspired the comments just after the proof of Lemma 2.3, we assume without loss of generality that x and Q are both inside L_0 and R is outside L'_{3m}, and that either: (i) y is outside L'_{3m}, or (ii) y is inside L_0.

First we assume that y is outside L'_{3m}. Harnack's inequality and Lemma 2.1 imply that $\max_{r\in R} P^y_r(Q) \le c\eta^m$.

$$\begin{aligned} I &= \int_R G(x,r)\frac{K(r,y)}{K(x,y)}\left(\int_Q G(q,r)\frac{K(q,y)}{K(r,y)}dq\right)\,dr \\ &\le \int_R G(x,r)\frac{K(r,y)}{K(x,y)}\left(C|Q|^{2/d}P^y_r(Q)\right)\,dr \\ &\le C|R|^{2/d}P_R\cdot|Q|^{2/d}\max_{r\in R}P^y_r(Q) \\ &\le C\eta^m|Q|^{2/d}|R|^{2/d}(P_Q+P_R), \end{aligned}$$

and

$$\begin{aligned} |II-III| &= \left|\int_Q\int_R G(x,q)\frac{K(q,y)}{K(x,y)}K(r,y)\left[\frac{G(r,q)}{K(q,y)}-\frac{G(x,r)}{K(x,y)}\right]dqdr\right| \\ &= \left|\int_Q\int_R G(x,q)\frac{K(q,y)}{K(x,y)}\cdot G(x,r)\frac{K(r,y)}{K(x,y)}\left[\frac{H(x,r)}{H(q,r)}-1\right]\,dqdr\right| \\ &\le C\eta^m\int_R G(x,r)\frac{K(r,y)}{K(x,y)}\,dr\int_Q G(x,q)\frac{K(q,y)}{K(x,y)}\,dq \\ &\le C\eta^m\cdot|Q|^{2/d}|R|^{2/d}P_Q\cdot P_R \\ &\le C_M e^{-C_M\rho(Q,R)}|Q|^{2/d}|R|^{2/d}(P_Q+P_R). \end{aligned}$$

The first inequality comes from Lemma 2.6 and the second from Lemma 2.2. Therefore, by (1.1),

$$|\mathrm{cov}(T_Q, T_R)| \le C_M e^{-C_M \rho(Q,R)} |Q|^{2/d} |R|^{2/d} (P_Q + P_R).$$

Next we assume y is inside L_0. This case is relatively easy to prove. The strong Markov property implies that

$$\begin{aligned} E^y_x T_Q T_R &\le E^y_x(E^y_{Z_{\tau_{Q^c}}} T_Q T_R I(\tau_{Q^c} < \tau)) \\ &\quad + E^y_x[T_R(E^y_{Z_{\tau_{Q^c}}} T_Q) I(\tau_{Q^c} < \tau)] \\ &\le C|Q|^{2/d}|R|^{2/d}\cdot\max_{q\in Q}P^y_q(R)\cdot P_Q \\ &\quad + C|Q|^{2/d}|R|^{2/d}\cdot\sqrt{P_R}\cdot P_Q \\ &\le C(\sqrt{\eta})^m|Q|^{2/d}|R|^{2/d}(P_Q+P_R) \\ &\le C_M e^{-C_M\rho(Q,R)}|Q|^{2/d}|R|^{2/d}(P_Q+P_R), \end{aligned}$$

the second inequality comes from the Schwartz inequality and (3.1). Therefore,

$$\begin{aligned}|\mathrm{cov}(T_Q, T_R)| &\le E_x^y T_Q E_x^y T_R + E_x^y T_Q T_R \\ &\le C_M e^{-C_M \rho(Q,R)} |Q|^{2/d} |R|^{2/d} (P_Q + P_R).\end{aligned}$$

□

This completes the proof of Theorem 2.1. Finally, we sketch an analytic proof of Theorem 1.1. Note that covariance formula (1.1) is true for any Greenian domain of any dimension. If ϕ is a conformal mapping from a simply connected planar domain D to the strip

$$S = (-\frac{\pi}{2} < \mathrm{Im}(z) < \frac{\pi}{2})$$

and $\psi(z) = e^{z+i\frac{\pi}{2}}$ conformally maps the strip to the upper-half plane R_+^2, we let $\Phi = \psi(\phi)$, $\Psi = \Phi^{-1}$, and $w' = \Phi(w)$. If $\rho(Q,R) \ge cm$, where c is an absolute constant that is bigger than c_2, the constant in Davis' Lemma 4.2 of [3], and m is an integer, then, together with this lemma, scaling implies that $\Phi(Q)$ is in L_0 and $\Phi(R)$ is outside L_{3m} and that furthermore neither $\Phi(x)$ nor $\Phi(y)$ belongs to that part of R_+^2 lying between L_0 and L_{3m}. We may assume that $\Phi(x)$ is inside L_0. Similarly, we have two cases to consider: (i) $\Phi(y)$ is inside L_0; (ii) $\Phi(y)$ is outside L_{3m}.

First we assume that y' is outside L_{3m}. The proof of (i) of Lemma 2.1 implies that $\max_{r\in R} P_r^y(Q) \le c\eta^m$. By Lemma 2.2,

$$\begin{aligned} I &= \int_R G(x,r)\frac{K(r,y)}{K(x,y)}\left(\int_Q G(q,r)\frac{K(q,y)}{K(r,y)}dq\right)\,dr \\ &\le \int_R G(x,r)\frac{K(r,y)}{K(x,y)}\left(C|Q|P_r^y(Q)\right)\,dr \\ &\le C|R|P_R\cdot|Q|\max_{r\in R}P_r^y(Q) \\ &\le C\eta^m|Q||R|(P_Q+P_R), \end{aligned}$$

and

$$\begin{aligned} |II-III| &= \left|\int_Q\int_R G(x,q)\frac{K(q,y)}{K(x,y)}K(r,y)\left[\frac{G(r,q)}{K(q,y)}-\frac{G(x,r)}{K(x,y)}\right]dqdr\right| \\ &= \Bigg|\int_{Q'}\int_{R'} G(x',q')\frac{K(q',y')}{K(x',y')}\cdot G(x',r')\frac{K(r',y')}{K(x',y')}\left[\frac{H(x',r')}{H(q',r')}-1\right] \\ &\qquad\qquad \cdot|\Psi'(q')|^2|\Psi'(r')|^2dq'dr'\Bigg| \\ &\le C\eta^m\int_{R'}G(x',r')\frac{K(r',y')}{K(x',y')}|\Psi'(r')|^2dr'\int_{Q'}G(x',q')\frac{K(q',y')}{K(x',y')}|\Psi'(q')|^2dq' \\ &\le C\eta^m\int_R G(x,r)\frac{K(r,y)}{K(x,y)}dr\int_Q G(x,q)\frac{K(q,y)}{K(x,y)}dq \\ &\le C\eta^m\cdot|Q||R|P_Q\cdot P_R \\ &\le Ce^{-c\rho(Q,R)}|Q||R|(P_Q+P_R). \end{aligned}$$

The first inequality comes from Lemma 2.6 and the third from Lemma 2.2. Therefore, by (1.1),

$$|\mathrm{cov}(T_Q, T_R)| \leq Ce^{-c\rho(Q,R)}|Q||R|(P_Q + P_R).$$

Next we assume y' is inside L_0. The proof of this case is the same as that of Theorem 1.2 and the only difference is that in the two-dimensional case the constants are absolute.

Acknowledgements

This paper is part of the author's doctoral dissertation, written under the guidance of Burgess Davis. I would like to thank him for his help.

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